2/6/20

Mixtures (Continued)

Part 1: Discrete Mixture of Discrete Distributions

Example: (Same as last time)

Bag of Dice

40% are 4-sided

60% are 6-sided

Let I = "indicator" r.v.

I	Pr
4	.4
6	.6

X = r.v. = the # rolled when 1 die is randomly ~~selected~~ selected

Note: $Pr(X=1) = Pr(X=1 \cap I=4) + Pr(X=1 \cap I=6)$

$= Pr(X=1 | I=4)\cdot Pr(I=4) + Pr(X=1 | I=6)\cdot Pr(I=6)$

$= (\frac{1}{4})\cdot(.4) + (\frac{1}{6})\cdot(.6) = .2$

$\therefore$ the distribution of X is

X	Pr
1	.2
2	.2
3	.2
4	.2
5	.1
6	.1

Unconditional Distribution of X

Consider $E[X]$ and $Var(X)$

$$E[X] = (1+2+3+4)(.2) + (5+6)(.1) = 3.1$$

$$E[X^2] = (1+4+9+16)(.2) + (25+36)(.1) = 12.1$$

$$Var(X) = 12.1 - (3.1)^2 = 2.49$$

Now consider the "parts" of the mixture

$(X^2 \mid I=4)$	$(X \mid I=4)$	Pr
1	1	all 1/4
4	2	
9	3	
16	4	

$(X^2 \mid I=6)$	$(X \mid I=6)$	Pr
1	1	all 1/6
4	2	
9	3	
16	4	
25	5	
36	6	

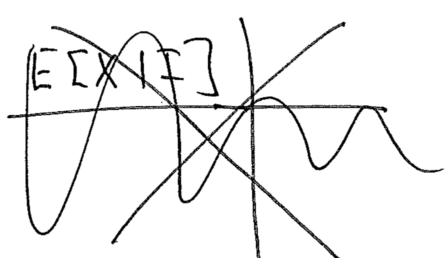
$$E[X \mid I=4] = (1+2+3+4)\cdot\tfrac{1}{4} = 2.5 = \tfrac{5}{2}$$

$$E[X \mid I=6] = (1+2+3+4+5+6)\cdot\tfrac{1}{6} = 3.5 = \tfrac{7}{2}$$

Remark on Notation: $(X \mid I=4)^2 = X^2 \mid I=4$

$$E[X^2 \mid I=4] = (1+4+9+16)\cdot\tfrac{1}{4} = \tfrac{30}{4}$$

$$E[X^2 \mid I=6] = (1+4+9+16+25+36)\cdot\tfrac{1}{6} = \tfrac{91}{6}$$



$$Var(X \mid I=4) = \tfrac{30}{4} - 2.5^2 = \tfrac{5}{4}$$

$$Var(X \mid I=6) = \tfrac{91}{6} - (3.5)^2 = \tfrac{35}{12}$$

Summarize:

I	$E[X \mid I]$	$Var(X \mid I)$	Pr
4	$5/2$	$5/4$	.4
6	$7/2$	$35/12$	.6

$$E\big[E[X \mid I]\big] = 5/2(.4) + \tfrac{7}{2}(.6) = 3.1 = E[X]$$

$$E\big[Var(X \mid I)\big] = \tfrac{5}{4}(.4) + \tfrac{35}{12}(.6) = 2.25 \quad (\neq 2.49 = Var(X))$$

$$+\ Var\big(E[X \mid I]\big) = \Big[\left(\tfrac{5}{2}\right)^2(.4) + \left(\tfrac{7}{2}\right)^2(.6)\Big] - \big[(3.1)^2\big] = 0.24$$

$$E[Var(X \mid I)] + Var(E[X \mid I]) = 2.25 + 0.24 = 2.49 = Var(X)$$

(Not a coincidence)

Facts:

1) $E[X] = E[E[X \mid I]]$ (Law of Total Expectation / Double Expectation)

2) $Var(X) = E[Var(X \mid I)] + Var(E[X \mid I])$
(Law of Total Variance)

For mixture problems, there are two ways to calculate variance.

1) $E[X] = E[E[X \mid I]]$

$E[X^2] = E[E[X^2 \mid I]]$

$\Rightarrow Var(X) = E[X^2] - (E[X])^2$

2) $Var(X) = E[Var(X \mid I)] + Var(E[X \mid I])$